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Transverse fivebranes in matrix theory

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Abstract: M-theory on the maximally supersymmetric plane wave background of eleven-dimensional supergravity admits spherical BPS transverse M5-branes with zero light-cone energy. We give direct evidence that the single M5-brane state corresponds to the trivial ($X = 0$) classical vacuum in the large N limit of the plane wave matrix theory. In particular, we show that the linear fluctuation spectrum of the spherical fivebrane matches exactly with the set of exactly protected excited states about the $X = 0$ vacuum in the matrix model. These states include geometrical fluctuations of the sphere, excitations of the worldvolume two-form field, and fermion excitations. In addition, we propose a description of multiple fivebrane states in terms of matrix model vacua. Finally, we discuss how to obtain the continuum D2/M2 and NS5/M5 theories on spheres from the matrix model. The matrix model can be viewed as a regularization for these theories.

Keywords: M-Theory, M(atrix) Theories, Penrose limit and pp-wave background.

Contents

1. Introduction

The matrix theory conjecture [1] states that the large N limit of the quantum mechanics obtained from the dimensional reduction of $d = 10$ SYM theory to 0+1 dimensions provides an exact description of light-cone M-theory in flat eleven-dimensional spacetime. There is now a large body of evidence supporting this conjecture (for a recent review, see [2]).

Perhaps the most basic test is that the matrix model should describe all of the usual objects expected in M-theory. For supergravitons [1], membranes [1], and longitudinal fivebranes (fivebranes extended in the light-cone directions) [3, 4], the matrix model description is by now very well known. Furthermore, it has been shown [1, 5, 6] that matrix theory correctly reproduces the low-energy interactions between arbitrary configurations of these objects expected from supergravity.

On the other hand, the matrix theory description of transverse fivebrane states (extended in the light-cone time direction and five transverse spatial directions) has remained somewhat mysterious. Notably, the charge corresponding to transverse fivebranes seems to be absent from the matrix theory supersymmetry algebra [3], though this does not rule out compact transverse fivebrane states which do not carry any net charge. There is some understanding of wrapped transverse fivebranes in matrix theory descriptions of M-theory on tori [7, 8]. However, we are not aware of any direct evidence for the appearance of transverse fivebrane degrees of freedom in non-compact matrix theory.

In this paper, we remedy this situation, providing detailed evidence that certain quantum states in matrix theory correspond to compact transverse fivebranes of M-theory.

Our demonstration is made possible by a number of simplifications which result from turning on background fields corresponding to the maximally supersymmetric plane wave of eleven-dimensional supergravity. On the gravity side, this background permits stable spherical transverse fivebrane states with zero light-cone energy [9], which should therefore appear as vacua of matrix theory (adding the appropriate operators to take into account coupling to the background). On the matrix theory side, the existence of a perturbative regime for certain values of the background parameter μp^+ [10] and a powerful supersymmetry algebra which protects energies and quantum numbers of certain states [11]–[13] allows us to extract exact information about the matrix theory spectrum in the M-theory limit. As a result, we are able to show in section 2 that the complete linear fluctuation spectrum of a single spherical transverse fivebrane is reproduced exactly as excited states about the trivial vacuum in the large N limit of the matrix model. This provides detailed support for the conjecture in [9] that the trivial vacuum of the matrix model corresponds to a spherical transverse fivebrane.

In section 3, we propose and present evidence for a description of arbitrary collections of concentric M5-branes together with concentric M2-branes in terms of vacua in the large N limit of matrix theory. We note that at finite N, the distinction between fivebranes and membranes is ambiguous. In section 4, we note a particular limit of the matrix model that can be used to describe the decoupled D2/M2 brane theories on a sphere. In section 5 we discuss similar limits which give the IIA NS5 brane little string theory on a fivesphere or the M5 brane theory on the fivesphere.

2. The fivebrane spectrum from matrix theory

Our starting point is the observation [9] that in the presence of a particular set of background fields, namely the maximally supersymmetric plane wave solution of eleven-dimensional supergravity, the classical action for the M5-brane has a zero light-cone energy solution corresponding to a stable spherical transverse M5-brane with radius

$$r^4 = \frac{\mu p^+}{6}\,, \tag{2.1}$$

as shown in detail in appendix B. We therefore expect that matrix theory with these background fields turned on should have a zero-energy vacuum state corresponding to the spherical fivebrane.

The relevant matrix model was described in [9]. The hamiltonian is

$$H = R \ \mathrm{Tr}\left(\frac{1}{2}\Pi_A^2 - \frac{1}{4}[X_A, X_B]^2 - \frac{1}{2}\Psi^\top \gamma^A [X_A, \Psi]\right) + \tag{2.2}$$
$$+ \frac{R}{2}\,\mathrm{Tr}\left(\sum_{i=1}^{3}\left(\frac{\mu}{3R}\right)^2 X_i^2 + \sum_{a=4}^{9}\left(\frac{\mu}{6R}\right)^2 X_a^2 + i\frac{\mu}{4R}\Psi^\top\gamma^{123}\Psi + i\frac{2\mu}{3R}\epsilon^{ijk}X_iX_jX_k\right).$$

This should describe M-theory on the maximally supersymmetric plane-wave background in the large N limit with fixed $p^+ = N/R$.

In [9] it was shown that this matrix model has a discrete set of classical supersymmetric vacua given by $X^i = \frac{\mu}{3R}J^i$ where J^i are the generators in an arbitrary N-dimensional reducible representation of SU(2). It is well known [14] that such matrix model configurations correspond to collections of membrane fuzzy-spheres with classical radii related to the dimensions N_i of the individual irreducible representations making up J^i by

$$r_i^2 = \frac{\mu^2}{9R^2}\frac{N_i^2 - 1}{4}\,.$$

These vacua are expected, since M-theory in the plane wave background also admits stable spherical membranes with radii $r = \mu p^+/6$ with zero light cone energy $-p_+ = 0$.

In [11], it was shown that all of these vacuum states must be exact quantum-mechanical vacua. Thus, the spherical fivebrane state should correspond to a quantum state given by some linear combination of these vacuum states. Since the classical fivebrane solution sits at the origin of the three dimensional space in which all of the membrane sphere solutions extend, a natural candidate for the fivebrane state is the trivial $X = 0$ vacuum for which J^i corresponds to N copies of the trivial representation of SU(2), as conjectured in [9]. The situation is similar to the one in [15]; in appendix D we make this connection a bit more precise.

2.1 Perturbation theory

The expansion parameter in perturbation theory for the matrix model about the $X = 0$ vacuum is [10]

$$\frac{NR^3}{\mu^3} = \frac{g_0^2 N}{\mu^3} = \frac{N^4}{(\mu p^+)^3}\,, \tag{2.3}$$

where g_0 is the zero brane coupling. At first sight, the $X = 0$ state looks very little like a spherical fivebrane. Classically, it resides at the origin for any value of μp^+, while the fivebrane is supposed to have radius $r \propto (\mu p^+)^{\frac{1}{4}}$. For weak coupling, it is straightforward to calculate that

$$\bar{r}^2 \equiv \langle 0|\frac{1}{N}\,\mathrm{Tr}(X_a^2)|0\rangle = \frac{18N^2}{\mu p^+}\left(1 + \mathcal{O}\left(\frac{N^4}{(\mu p^+)^3}\right)\right). \tag{2.4}$$

Thus, for fixed N in the perturbative regime the size of the $X = 0$ state actually decreases as $N^2/(\mu p^+)$ when μp^+ becomes large rather than increasing as $(\mu p^+)^{1/4}$. On the other hand, for the validity of the matrix theory conjecture, the classical expression (2.1) for the fivebrane radius need only be reproduced in the limit of large N with fixed μ and

$p^+ = N/R$. In this limit, the effective coupling (2.3) always becomes large, so perturbation theory is inapplicable. Thus, it is possible that there is a transition from (2.4) to (2.1) when we take the large N limit. Intriguingly, both (2.1) and (2.4) become of the same order of magnitude when the coupling (2.3) is of order one.

2.2 Protected quantities

From the discussion in the previous section, it appears that any fivebrane-like properties of the $X = 0$ vacuum will emerge only as strong coupling effects in the matrix model, and would therefore be extremely difficult to observe directly. Fortunately, as shown in [11], the SU(4|2) symmetry algebra of the matrix model implies that certain physical quantities are exactly protected for all values of $\mu > 0$. For any value of N, these may be calculated in the $\mu \to \infty$ limit where the theory is free and then extrapolated to any desired value of $\mu > 0$. In this way, it is possible to obtain reliable information about the matrix model at a given value of μp^+ even in the large N limit where the theory is strongly coupled.

The protected quantities that we will be interested in are the energies and quantum numbers of certain excited states about the $X = 0$ vacuum. As discussed in [11, 13], physical states of the matrix model must lie in representations of SU(4|2) which are comprised of finite collections of representations of the bosonic subalgebra SO(6) $\times$ SO(3) $\times$ Energy. Among the physically allowable SU(4|2) representations, there are certain BPS representations which are exactly protected, that is, the energy (in units of μ) and SO(6) $\times$ SO(3) state content cannot change as μ is varied.[1] Thus, any such representation present at $\mu = \infty$ (where the exact spectrum was calculated in [10]) must be in the spectrum for any value of $\mu > 0$.

It turns out that the spectrum of excitations about the $X = 0$ vacuum contains infinite towers of these exactly protected representations [11]. To describe these, we recall that at $\mu = \infty$, the theory becomes quadratic, with hamiltonian

$$H_2 = \mu \,\mathrm{Tr}\left(\frac{1}{3}A_i^\dagger A_i + \frac{1}{6}A_a^\dagger A_a + \frac{1}{4}\psi^{\dagger I\alpha}\psi_{I\alpha}\right),$$

and the spectrum of excitations about the $X = 0$ vacuum are generated by traces of products of the matrix creation operators

$$\psi_{I\alpha}^\dagger\,, \qquad A_i^\dagger = \sqrt{\frac{\mu}{6R}}X^i - i\sqrt{\frac{3R}{2\mu}}\Pi^i\,, \qquad A_a^\dagger = \sqrt{\frac{\mu}{12R}}X^a - i\sqrt{\frac{3R}{\mu}}\Pi^a\,.$$

Among the states containing only a single trace, we have a single tower of exactly protected SU(4|2) multiplets, described by primary states

$$C^{a_1\cdots a_n}\,\mathrm{Tr}(A_{a_1}^\dagger\cdots A_{a_n}^\dagger)|0\rangle \tag{2.5}$$

plus states obtained from these by acting with supersymmetry generators. Here $C^{a_1\cdots a_n}$ is a completely symmetric, traceless tensor of SO(6). The remaining exactly protected primary states are identical in form but have a U(N) index structure involving more than one trace. In appendix A we give a simple argument for why these states are protected.

[1]Note that only certain BPS multiplets are exactly protected, since others may combine and form non-BPS multiplets.

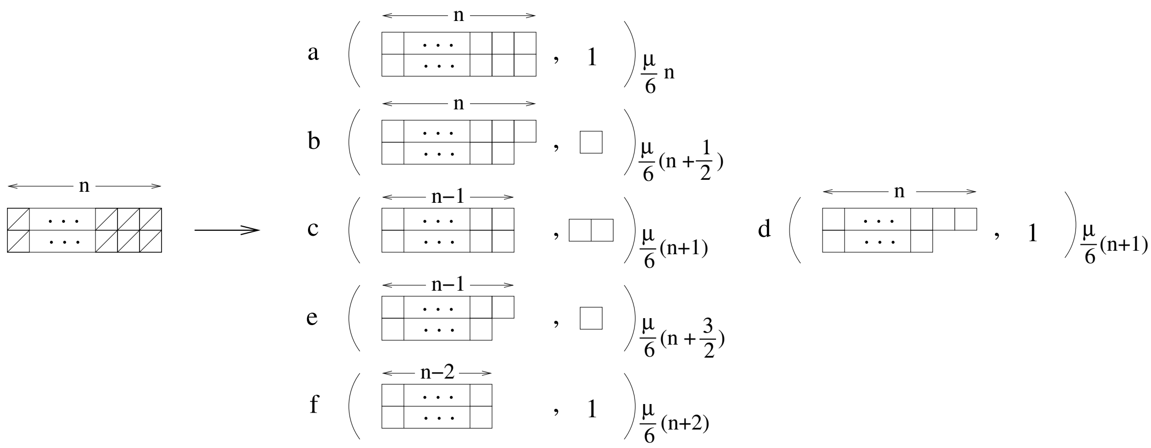

Figure 1: SU(4) $\times$ SU(2) tableau and energies for exactly protected excitations about the $X = 0$ vacuum.

The complete set of SO(6) $\times$ SO(3) representations that descend from these primary states is displayed in figure 1. This shows the spectrum of single-trace exactly protected multiplets about the $X = 0$ vacuum in the large N limit defining M-theory.[2] If the $X = 0$ vacuum represents a spherical M5-brane, these states should be among the excitations of the fivebrane.

In fact, the spectrum of exactly protected matrix theory states given in figure 1 matches precisely with the linear fluctuation spectrum of the spherical fivebrane solution!

As shown in appendix B, geometrical fluctuations of the fivebrane in the radial and x^- directions are described by the representations **a** and **f** in figure 1, geometrical fluctuations in the three transverse directions give modes which make up the representation **c**, two-form fluctuations yield the remaining tower of bosonic states **d**, and the representations **b** and **e** are excitations of the worldvolume fermions.

In a similar way, the exactly protected multi-trace states will match with protected fivebrane fluctuations containing several quanta.

Thus, the exactly protected excited states above the $X = 0$ vacuum precisely correspond to the fluctuations of the spherical transverse M5-brane in the plane wave background. This represents compelling evidence that the $X = 0$ vacuum of the matrix model does indeed describe the spherical transverse fivebrane.

3. Multiple fivebranes

In addition to the single fivebrane state we have discussed, M-theory on the maximally supersymmetric plane wave should contain states with concentric fivebranes of arbitrary radii, as is the case for the spherical membranes. We will now propose a matrix model description of these and then present evidence for the proposal.

[2] For finite N, there are identities which relate certain single trace operators with multiple trace operators so we have a truncated version of the spectrum.

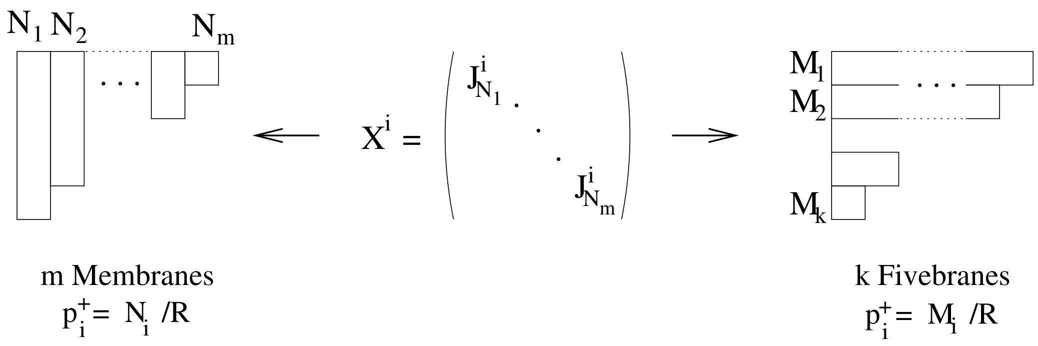

Figure 2: Dual membrane and fivebrane interpretations for general vacua.

At finite N, there is a natural one-to-one correspondence between vacua of the matrix model and ways of distributing N units of momenta between any number of membranes. A vacuum corresponding to a partition $N = N_1 + \cdots + N_m$ (where N_i label the sizes of the associated SU(2) irreps) has a membrane interpretation as concentric fuzzy spheres with radii proportional to the individual momenta N_i/R. On the other hand, we expect equally many fivebrane states, since states with N units of momentum divided between k fivebranes would also be labelled by partitions of N.[3]

Since we have already associated all the vacua with membrane states, it is evident that the distinction between membrane states and fivebrane states must be somewhat ambiguous at finite N. As an example, we note that the state corresponding to a partition $N = 1 + \cdots + 1$ would be given a membrane interpretation as N membranes each carrying one unit of momentum, but this is precisely the state that we have associated with a single fivebrane (carrying N units of momentum).

In fact, there is a natural dual fivebrane interpretation for each of the vacuum states. To describe this, note that any partition of N may be represented by a Young diagram whose column lengths are the elements in the partition. In the membrane interpretation, such a diagram would correspond to a state with one membrane for each column with the number of boxes in the column corresponding to the number of units of momentum. In the dual fivebrane interpretation, it is the rows of the Young diagram that correspond to the individual fivebranes, with the row lengths corresponding to the number of units of momentum carried by each fivebrane, as shown in figure 2.

With this interpretation, it is clear that the number of fivebranes is equal to the size of the largest irreducible representation, while the momentum M_n carried by the nth fivebrane is equal to the number of irreducible representations with size greater than or equal to n.

Since we now have both membrane and fivebrane interpretations for each of the vacua, an obvious question is which of these interpretations is more appropriate.[4] In general, the answer will depend on the values of (μp^+), N and the parameters describing the vacuum of interest. Given any fixed choice of vacuum, the membrane interpretation will be correct

[3] As discussed below, we might also have vacua which include both membranes and fivebranes.

[4] In principle, we may answer this question by finding the expectation value of various operators corresponding to multipole moments of the charge distributions to determine the geometry of the state.

for small values of the effective coupling (i.e. sufficiently small values of $1/(\mu p^+)$), since the theory becomes free in this limit and the classical geometry corresponding to concentric fuzzy spheres will not receive quantum corrections. On the other hand, our single fivebrane example suggests that if the number of representations is large enough so that the coupling is large, then the fivebrane interpretation should be appropriate. For general intermediate values of the parameters, the identification as membranes or fivebranes is likely ambiguous.

The situation is clearer in the large N limit defining M-theory. Here, to define a state with a fixed number of membranes with various momenta, we keep the number of irreducible representations and the ratio of their sizes fixed as we take the large N limit. The sizes N_i of the individual representations thus go to infinity and the momentum fractions N_i/N carried by the various membranes become continuous parameters. On the other hand, to define states with fixed numbers of fivebranes, we keep the sizes of representations fixed and take the number of representations to infinity. In this case, the number of fivebranes corresponds to the maximum representation size, and the momentum fraction carried by the nth fivebrane is M_n/N where M_n is the length of the nth row in the Young diagram.

In terms of the Young diagrams of figure 2, membrane/fivebrane states correspond to diagrams with a fixed number of columns/rows in the large N limit. One may also consider more general limits in which both the number of rows and the number of columns become infinite. It is natural to identify states with m infinite columns and k infinite rows with configurations including both m concentric membranes and k concentric fivebranes. On the other hand, it is not clear how to interpret states where the number of infinite rows or columns becomes infinite (e.g. the large N limit of the diagram with rows of length $1, 2, 3, \ldots N$ or the square diagram with N rows of length N).

Before proceeding to give evidence for this proposal, we note that the description of multiple fivebrane states is very different from the usual classical picture in which different blocks in block-diagonal matrices correspond to different objects. For example, it is not true that taking a block diagonal configuration where each block represents the classical matrix for a single fivebrane leads to a multiple fivebrane state.

3.1 Evidence for the proposal

Given this explicit proposal for the identification of multiple M5-branes states with matrix model vacua in the large N limit, we would now like to see what evidence supports it.

Firstly, with our construction, there is a one-to-one correspondence between the ways of dividing N units of momenta between k fivebranes and matrix model vacua involving irreducible representations of size k.

Further evidence comes by considering the exactly protected states about the various vacua, as we have done for the single fivebrane case. Consider first the M-theory states with k coincident spherical fivebranes. The corresponding vacua are those with N copies of the k dimensional irreducible representation in the large N limit. From the results of [10, 11], it is straightforward to verify that the exactly protected excited states above these vacua are the same for any value of k (at any fixed N or in the large N limit).[5]

[5]Here, we mean states which are exactly protected by SU(4|2) representation theory. As we discuss

This is consistent with the idea that the k coincident fivebranes form a bound state whose protected fluctuations are the same as for a single fivebrane.

Next consider the general vacua corresponding to partitions

$$(1, \ldots, 1, 2, \ldots, 2, \ldots, k, \ldots, k)$$

in the limit where the number of each representation goes to infinity with fixed ratios. Here, the single-trace exactly protected states form k copies of the single-trace exactly protected spectrum of a single fivebrane. This fits in well with our interpretation of such a state as k concentric fivebranes at different radii, since the k copies may be interpreted as independent fluctuations of the k individual fivebranes. In the case where the fraction of representations of size n drops to zero for some $n < k$, we lose one set of protected states, and this may be interpreted as the nth largest fivebrane shrinking to form a bound state with the $(n+1)$st largest fivebrane.

It is important to note that the calculated fluctuation spectrum in the appendix did not depend on the radius of the fivebrane. This is the reason that we simply have k copies of the single-fivebrane fluctuation spectrum for the matrix model states corresponding to k fivebranes at arbitrary radii. For the k-coincident fivebrane state, it is possible to obtain some evidence that the corresponding matrix model state has the correct physical size by repeating the suggestive calculation of section 2. For the state with m copies of the k dimensional representation ($N = mk$) we find

$$\bar{r}^2 \equiv \left\langle \frac{1}{N} \operatorname{Tr}(X_a^2) \right\rangle = 18 \left(\frac{R}{\mu} \right) m \left(1 + \mathcal{O}\left(m \frac{R^3}{\mu^3} \right) \right).$$

As m is increased with fixed $\mu p^+ = \mu mk/R$, both the size of the state and the size of the perturbative corrections grow, and the size matches with the expected value $\bar{r} \approx (\mu p^+/k)^{1/4}$ just as the effective coupling becomes of order 1. Of course, there is no reason to expect that such a calculation should give the desired radius (which we only require to be reproduced for $m \to \infty$) except that it worked for the single fivebrane case.

3.2 Chiral operators for multiple membranes and fivebranes

To give a final piece of evidence that the proposed description of fivebranes is correct, we consider in some detail the matrix model vacua with m copies of the k dimensional irreducible representation. Depending on the parameters, this vacuum may describe m coincident membranes (e.g. if $k \to \infty$) or k coincident fivebranes (e.g. for $m \to \infty$). In general, the excitations about this vacuum state (at large μ) are generated by $m \times m$ matrix oscillators in the SU(4|2) representations corresponding to single-column supertableau with $2, 4, \ldots, 2k$ boxes.

The exactly protected single trace states arise from a trace containing up to m two-box oscillators (traces of more than this number are not independent). For $m \to \infty$, we obtain the complete spectrum of figure 1, and as we have discussed, these states correspond

below, there may be other states whose energies are protected for other reasons even though group theory does not forbid them from receiving energy shifts.

to protected fluctuations of the bound state of k fivebranes. On the other hand, in the membrane limit $k \to \infty$ with fixed m, we still have a truncated version of the spectrum in figure 1, and we may ask how this should be interpreted in the membrane picture.

To answer this, recall that the SO(8) superconformal theory describing the low-energy theory of m coincident M2-branes in M-theory has chiral operators in symmetric traceless representations of SO(8) with up to m indices (analogous to the chiral operators of $N = 4$ SYM with in symmetric traceless representations of SO(6).) These operators should correspond to states of the membrane theory on $S^2 \times R$. In the plane wave background, the theory living on the spherical membranes only has an SO(6) subset of the SO(8) R-symmetry preserved. Therefore, if states corresponding to the chiral operators survive for spherical M2-branes in this background, we would expect them to show up as states in symmetric traceless representations of the SO(6) with up to m indices. The exactly protected states we have discussed have precisely these quantum numbers and this cutoff (and also the right relationship between energy and quantum numbers), so it is natural to conclude that they correspond to the chiral operators.

Given that we see evidence of the chiral operators of the SO(8) superconformal theory, we can now ask whether the k-fivebrane states also have excitations corresponding to the chiral operators of the $(0, 2)$ superconformal theory. In this case, the chiral primary operators lie in symmetric traceless representations of the SO(5) R-symmetry with up to k indices. For spherical fivebranes on the plane wave background, only an SO(3) subgroup of this SO(5) is preserved, so in this case, we expect protected states in SU(2) representations with spins up to k. In fact, there is a set of single trace states in the spectrum with precisely these quantum numbers, obtained by taking traces of the single oscillators mentioned above.[6]

We would thus like to associate these single oscillator states with chiral operators of the $(0, 2)$ theory, however it is not clear a priori that they should be protected. These multiplets are BPS, however as discussed in [10], BPS SU(4|2) multiplets corresponding to supertableau with more than three rows (or more than one box in the third row) have the possibility of combining with certain other BPS multiplets to receive energy shifts. In fact, there are examples [10, 12] of states in precisely these single-column representations which receive energy shifts in perturbation theory.

On the other hand, there is some evidence that the particular states in question do not receive energy shifts. For the case $m = 1$, it was argued in [10] that the 4-box and 6-box single oscillator states cannot receive corrections to any order in perturbation theory, while the 8-box and 10-box states are protected at least to leading order in perturbation theory. A more general argument is that the quadratic fluctuation spectrum of a spherical membrane in this background gives precisely the set of SU(4|2) representations corresponding to supertableaux with $2, 4, 6 \ldots$ vertical boxes. By analogy with the fivebrane case, we would then expect protected states in the matrix model which could be identified with these states in the large N limit. The natural candidates for such protected states are the

[6] The oscillator in the SU(4|2) representation with $2n$ vertical boxes includes "primary" states with purely SU(2) quantum numbers for spin n plus states obtained from this by acting with supercharges.

single-oscillator states (these are the only choice for $m = 1$). In other words, we expect that the states that match with the membrane fluctuation spectrum in the large k limit should be protected, and that these states should correspond to the chiral 5-brane operators in the large m limit with fixed k.

4. The matrix model as a regularization of the D2/M2 brane theories

Since the D0 brane matrix model expanded around vacua with large SU(2) representations looks like the D2 theory on a fuzzy sphere it is natural to ask whether we can think of this matrix model as a regularization of the D2 theory. It is interesting as a regularization because it preserves the 16 supersymmetries of the D2 brane theory. These supercharges anticommute to the hamiltonian plus rotations of the sphere. In the limit that the size of the sphere is large these rotations become translations and rotations of the spatial plane. On the other hand, a lattice regularization will break more supersymmetries (see [16] for a discussion). In the next subsection we discuss this point in more detail.

4.1 Decoupling limits

Consider the matrix model expanded around the k membrane vacuum where we have k copies of the N dimensional representation of SU(2). We find that the theory looks like a fuzzy sphere of radius μ^{-1} with non-commutativity parameter and coupling constant

$$\theta = \frac{1}{\mu^2 N}, \qquad g^2_{2YM} = \frac{g^2_{0YM}}{\mu^2 N}. \tag{4.1}$$

We are interested in the limit $N \to \infty$ keeping the two dimensional gauge coupling g^2_{2YM} and μ fixed. In this limit, the noncommutativity parameter vanishes and we obtain a continuum theory on a two sphere with sixteen supercharges and action

$$\begin{aligned} S = \frac{1}{g^2_{YM}} \int dt \frac{d\Omega}{\mu^2}\, \mathrm{tr} \Bigg(& -\frac{1}{4} F^{\mu\nu} F_{\mu\nu} - \frac{1}{2}(D_\mu X^a)^2 - \frac{1}{2}(D_\mu \phi)^2 + \frac{i}{2}\Psi^\dagger D_0 \Psi - \\ & -\frac{i}{2}\epsilon^{ijk}\Psi^\dagger\gamma^i x^j D_k \Psi + \frac{1}{2}\Psi^\dagger\gamma^i x^i[\phi,\Psi] + \frac{1}{2}\Psi^\dagger\gamma^a[X^a,\Psi] + \frac{1}{4}[X_a,X_b]^2 + \\ & + \frac{1}{2}[\phi,X^a]^2 - \frac{\mu^2}{8}X^a X^a - \frac{\mu^2}{2}\phi^2 - \frac{3i\mu}{8}\Psi^\dagger\gamma^{123}\Psi + \frac{\mu}{2}\phi\epsilon^{ijk}x^i F_{jk} \Bigg), \end{aligned} \tag{4.2}$$

where trace and the commutators are those of $k \times k$ matrices and $x_i^2 = 1$. The first and second line contain the terms present in the usual $2+1$ dimensional SYM theory. The third line contains mass terms for the scalars and fermions plus an extra ϕF interaction. The radius of the sphere is proportional to μ^{-1}. The derivation of this lagrangian, as well as the supersymmetry transformations, can be found in appendix C.

In the $\mu \to 0$ limit this action becomes the action of $2+1$ Yang Mills in flat space and the supersymmetry of the theory becomes the supersymmetry of the flat space $2+1$ Yang Mills theory.

Similarly we could consider the $g^2_{YM} \to \infty$ limit keeping μ fixed which would give us the superconformal theory associated to M2 branes on $S^2 \times R$.

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The ϕ scalar in (4.2) is basically associated to the radial direction. We can imagine adding a magnetic flux over S^2. If the theory we start with is the $\mathrm{U}(Nk)$ matrix model then adding a magnetic flux on S^2 is equivalent to starting with the $\mathrm{U}(Nk+n)$ matrix model. In this case, the final theory will contain an additional flux $\int_{S^2} \mathrm{Tr}[F] = n$. As a result, the vacuum of the theory (4.2) is given by a $\phi \sim n\mu/k$. Note that if we start with an $\mathrm{SU}(Nk)$ matrix model we get a two brane theory of the form (4.2) with a gauge group $\mathrm{U}(k)$ but with the zero modes on the sphere of the center of mass $\mathrm{U}(1)$ removed.

In the theory (4.2), we can also consider vacua where $\int_{S^2} \mathrm{Tr}\, F = 0$ but where F is diagonal with ϕ similarly diagonal and different entries along the diagonal. For example if $k = 2$ we can choose $F \sim \mu\phi \sim \mathrm{diag}(n, -n)$. This vacuum has zero energy and can be thought of as coming from a representation of $\mathrm{SU}(2)$ containing two irreducible representations, one of dimension $N + n$ and one of dimension $N - n$. Clearly this configuration should be included in the path integral of the theory (4.2). In fact it is possible to estimate the tunnelling amplitude in the matrix model between this vacuum and the vacuum with two representations of equal size. In supersymmetric quantum mechanics the tunnelling amplitude between two supersymmetric vacua can be estimated as the difference in superpotential between them. In our case the superpotential in question has the form $W \sim \frac{1}{g_0^2} \mathrm{Tr}[\epsilon_{ijk} X^i X^j X^k + \mu X^i X^i]$ so that the difference in superpotential is proportional to the difference in the trace of the second casimir of the $\mathrm{SU}(2)$ representation that defines the vacuum. In particular, the superpotential difference between a vacuum with two representations of size N and the one with representations of sizes $N + 1$, $N - 1$ is of the order of N/g_0^2. In our limit we are keeping this constant so that the tunnelling amplitude is not suppressed. In other words, we cannot isolate a particular vacuum of the matrix model, but the vacua we can tunnel into have a perfectly good interpretation from the D2 brane point of view and should be included in the definition of the theory. Vacua where the difference in dimension of the representations goes to infinity as N goes to infinity are very far away and the matrix model cannot tunnel to them in finite time.

In order to understand how the action (4.2) relates to an M2-brane theory, it is useful to take a $\mathrm{U}(1)$ gauge group in (4.2). We can then dualize the $\mathrm{U}(1)$ field strength. Due to ϕF coupling in (4.2) the dualization is slightly different than the one for the flat space D2 action. Namely the dual scalar is defined by

$$d\varphi = \frac{1}{g^2}(*F + \mu\phi dt)\,. \tag{4.3}$$

Then the equations of motion for ϕ and φ can be rewritten as

$$\nabla^2\varphi + \frac{\mu}{g^2}\partial_0\phi = 0\,, \qquad \nabla^2\phi - \mu g^2\partial_0\varphi = 0\,. \tag{4.4}$$

These can be viewed as the equations of motion for two of the transverse scalars of the $M2$ theory. More precisely, let us denote by Z a complex combination of two of those scalars. Then take a configuration with $Z_0 = g^2 e^{i\mu t}$ which is classically rotating. Then we can expand to first order $Z = (g^2 + \phi)e^{i\mu t + i\varphi}$. The equation of motion for Z is of course a harmonic oscillator equation with frequency μ. This leads to the above equations (4.4) for

ϕ, φ. In the large g^2 limit, we see that we get the M2 theory expanded around a state with very high angular momentum in an SO(2) subgroup of SO(8). For this reason we only see an explicitly SO(6) symmetry in (4.2). For the nonabelian case, one would be tempted to say that the theory we get is simply the usual M2 theory since the angular momentum can be carried only by the overall U(1).

4.2 Gravity perspective

Now we analyze these decoupling limits from the gravity perspective. The main point of this exercise is to learn how to do it since for the NS5/M5 case we will only have this gravity description.

We wish to take the limit in which the radius of the M2 brane becomes very large in Planck units. This is achieved by taking large values of $r = \mu p^+$. It is convenient to rewrite the relevant terms in the metric around this large radius two sphere that the brane is wrapping as

$$ds^2 = -2dx^+dx^- - \mu^2 r^2 (dx^+)^2 + r^2 d\Omega_2^2 + \cdots = \frac{1}{r^2\mu^2}(dx^-)^2 - \mu^2 r^2 (d\tilde{x}^+)^2 + r^2 d\Omega_2^2 + \cdots , \tag{4.5}$$

where we defined the variable $\tilde{x}^+ = x^+ + x^-/(r\mu)^2$ and the dots indicate terms that are not needed for our discussion. We will be interested in keeping μ fixed and taking $r \to \infty$.

The physical compactification radius of the x^- direction is $\tilde{R} = R/(\mu r)$.[7] The D2-brane theory is then characterized by a dimensionful coupling given by the usual relation $g_{YM} = \tilde{R}/l_p^{3/2}$. In the worldvolume theory, it is convenient to rescale the metric so that the sphere has unit radius, after which the theory may be characterized by the dimensionless coupling

$$g_{YM} r^{1/2} = \tilde{R}\sqrt{r} = \frac{R}{r^{1/2}\mu} \tag{4.6}$$

which we keep fixed in the limit (we have set $l_p = 1$).

We see that in order for this to be finite as $r \to \infty$ with fixed μ, we need that $R^2/r \sim R^3/N \sim g_0^2/N$ is finite in agreement with (4.1), where we have used $r \sim \mu p^+ = \mu N/R$.

5. The matrix model as a regularization of the NS5/M5 theories

In this case we can only use the supergravity reasoning to guide us on what the limit should be since we cannot derive the NS5 (or a non-commutative version [17] for finite N) at weak coupling from the matrix model. Now the relation between the radius and the momentum is (2.1). We can again write the metric in a form similar to (4.5). If we scale R in a suitable way and $N \to \infty$ we expect to get an NS5 brane theory on a fivesphere of radius r.

The NS5 brane theory, the so called little string theory, is characterized by the string tension $1/\alpha' = \tilde{R}/l_p^3$. In this case, the dimensionless quantity that we would like to hold

[7] Note that when we shift x^- by its period we also shift x^+ but the shift in x^+ will go to zero in the limit that we are taking when we keep the two dimensional gauge coupling of the D2 theory fixed.

fixed is the tension in units of the radius of the sphere, given by

$$\frac{r^2}{\alpha'} = \tilde{R} r^2 = \frac{R}{\mu r} r^2 = \frac{\left(g_0^2 N\right)^{1/4}}{\mu^{3/4}}\,, \tag{5.1}$$

where we used (2.1).[8] Since we are holding μ fixed, the NS5 limit corresponds to the 't Hooft limit of the matrix model. The strings of the little string theory are the usual 't Hooft strings. This makes this discussion be very similar to the discussion of Polchinski and Strassler [15] for D3 branes.

The limit that defines the M5 theory is the limit where, in addition, we take (5.1) to infinity. This is a definition of the M5 brane theory which preserves 16 of its supersymmetries in an explicit way.

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A. Protected states in the matrix theory spectrum

In this appendix we present a simple argument, without much group theory, for why some states are protected. A full discussion can be found in [11]. The idea is to define an index, as in [18], to which only the states in question contribute at weak coupling. The index will be independent of the coupling so its value can be computed at weak coupling. Below we explain this in more detail.

It will be important for us to concentrate on the bosonic symmetry generators that we can simultaneously diagonalize, H, M^{12}, M^{45}, M^{67}, M^{89}. These are the hamiltonian of the matrix model, a U(1) subgroup of the SU(2) symmetry rotating the first three coordinates and a $\mathrm{U}(1)^3$ subgroup of the SO(6) rotating the other six coordinates. The supercharges transform as spinors under the rotation groups and they raise or lower the value of the matrix model energy. It is convenient to focus on the energy raising supercharge $Q^\dagger = Q^\dagger_{-+++}$, where the subindices indicate the transformation properties under $M^{12}, M^{45}, M^{67}, M^{89}$. This supercharge and its adjoint obey the anticommutation relation

$$\{Q, Q^\dagger\} = H - \frac{1}{3}M^{12} - \frac{1}{6}(M^{45} + M^{67} + M^{89}) \equiv \tilde{H}\,, \tag{A.1}$$

where we have defined $\tilde{H}$. Note that $\tilde{H}$ has a non-negative spectrum. We should also note that $\tilde{H}$ commutes with $Q, Q^\dagger$. We could then consider the index $\mathrm{Tr}[(-1)^F e^{-\beta\tilde{H}}]$ which receives contributions only from BPS states with $\tilde{H} = 0$. We find however that

[8] Note that we may also write this as $\lambda^{1/4}$ where $\lambda = N^4/(\mu p^+)^3$ was the parameter (2.3) controlling perturbation theory about the $X = 0$ vacuum.

in perturbation theory there are many bosonic and fermionic states with $\tilde{H} = 0$. For this reason it is convenient to introduce another operator J which commutes with $Q, Q^\dagger$ (and therefore with $\tilde{H}$) and restrict the index to subspaces with definite values of J. It is convenient to pick $J = H - \frac{1}{6}M^{89}$. Then one can check that all states with $\tilde{H} = 0$ have $J \geq 0$, and the only perturbative states with $J = 0$ are the ones created by an oscillator mode in the 89 plane with positive angular momentum M^{89}. In the notation of section 2 this is $A_8^\dagger + iA_9^\dagger$. Since all states created by this mode are bosonic we do not have any cancellations in the index. So all BPS states with $\tilde{H} = 0$ and $J = 0$ are exactly protected in the full theory. Of course we should only consider gauge invariant states. Once we take into account the SO(6) symmetry we see that we recover the result stated in (2.5) and used in section 2.

B. Linear fluctuation spectrum of the fivebrane

In this appendix, we compute the linear fluctuation spectrum for the spherical vacuum state of an M-theory fivebrane in the maximally supersymmetric plane wave background of eleven-dimensional supergravity, given by

$$\begin{aligned} ds^2 &= -2dx^+dx^- + \sum_{A=1}^{9} dx^A dx^A - \left(\sum_{i=1}^{3} \frac{\mu^2}{9} x^i x^i + \sum_{a=4}^{9} \frac{\mu^2}{36} x^a x^a \right) dx^+ dx^+ \\ F_{123+} &= \mu \,. \end{aligned} \tag{B.1}$$

Ignoring for now the worldvolume two-form field and fermions, we find that the light-cone gauge hamiltonian for a fivebrane in this background is given by

$$\begin{aligned} \mathcal{H} &= \int d^5\sigma \frac{1}{2p^+}\{P_A^2 + |g_{AB}|\} - \frac{p^+}{2} g_{++}(X) - i_+ C^{(6)} \\ &= \frac{1}{2p^+}\left(P_A^2 + \frac{1}{5!}\{X^{A_1}, \ldots, X^{A_p}\}\{X^{A_1}, \ldots, X^{A_p}\} \right) + \\ &\quad + \frac{p^+}{2}\left(\left(\frac{\mu}{6}\right)^2 X^a X^a + \left(\frac{\mu}{3}\right)^2 X^i X^i \right) + \frac{\mu}{6!}\epsilon_{a_1 \cdots a_6} X^{a_1}\{X^{a_2}, \ldots, X^{a_6}\}\,, \end{aligned}$$

where we define

$$\{A_1, \ldots, A_5\} = \epsilon^{\alpha_1, \ldots, \alpha_5} \partial_{\alpha_1} A_1 \cdots \partial_{\alpha_5} A_5$$

and we have chosen worldvolume coordinates σ_α such that $d\sigma_1 \cdots d\sigma_5$ is the volume element.

To see that the spherical fivebrane represents a zero-energy solution, note that setting $X^i = 0$, the potential may be written as a perfect square,

$$V_{X^a} = \frac{1}{2p^+}\left(\frac{\mu p^+}{6} X^a + \frac{1}{5!}\epsilon^{a a_1 \cdots a_5}\{X^{a_1}, \ldots, X^{a_5}\} \right)^2 . \tag{B.2}$$

It is convenient to define functions $x^a(\sigma)$ which map the worldvolume into a target space unit sphere. Then $x^a x^a = 1$ and

$$\{x^{a_1}, \ldots, x^{a_5}\} = \epsilon^{a_1 \cdots a_6} x^{a_6} \,. \tag{B.3}$$

From this relation, it is clear that

$$X^a = rx^a$$

gives a zero-energy solution if the sphere radius satisfies

$$r^4 = \frac{\mu p^+}{6}.$$

We would now like to expand the potential about this classical solution and determine the quadratic fluctuation spectrum.

X^a fluctuations. Setting $X^a = rx^a + Y^a$, we find from (B.2) that the quadratic potential for the X^a fluctuations is

$$V_2^{X^a} = \left(\frac{p^+}{2}\right)\left(\frac{\mu}{6}\right)^2 \left(Y^a + \frac{1}{24}\epsilon^{aa_1\cdots a_5}\{x^{a_1},\ldots,x^{a_4},Y^{a_5}\}\right)^2 .$$

Normal modes will be solutions of the eigenvalue equation

$$\mathcal{L}_{ab}Y_b \equiv \frac{1}{24}\epsilon^{aa_1\cdots a_5}\{x^{a_1},\ldots,x^{a_4},Y^{a_5}\} = \lambda Y^a \tag{B.4}$$

with with masses given by

$$M^2 = \left(\frac{\mu}{6}(1+\lambda)\right)^2 .$$

Here, $\mathcal{L}_{ab}$ are generators of SO(6), so the eigenvectors will be (vector) spherical harmonics of SO(6) given explicitly by

$$\begin{aligned}
Y_l^a &= S_{aa_1\cdots a_l}x^{a_1}\cdots x^{a_l} && M = \frac{\mu}{6}(l+1)\\
\tilde{Y}_l^a &= x^a\tilde{S}_{a_1\cdots a_{l-1}}x^{a_1}\cdots x^{a_{l-1}} - \frac{l}{2l+2}\tilde{S}_{aa_1\cdots a_{l-2}}x^{a_1}\cdots x^{a_{l-2}} && M = \frac{\mu}{6}(l+3)\\
\hat{Y}_l^a &= A^a_{a_1\cdots a_l}x^{a_1}\cdots x^{a_l} && M = 0\,.
\end{aligned}$$

Here the tensors S and $\tilde{S}$ are symmetric and traceless and A is an SO(6) tensor with indices a and a_1 antisymmetric. These correspond to the three irreducible representations in the tensor product of the vector and l-index symmetric traceless representations of SO(6), with Dynkin labels $(0, l+1, 0)$, $(0, l-1, 0)$ and $(1, l-1, 1)$ in the order listed above. The representations and energies of the modes Y and $\tilde{Y}$ match exactly with the representations **a** and **f** of figure 1, with $n = l+1$. The zero-modes $\hat{Y}$ are non-physical since they are fluctuations in the gauge orbit directions under the gauge group of volume-preserving diffeomorphisms.

X^i fluctuations. For the X^i modes, we find that the quadratic action is given by

$$S_2^{X^i} = \frac{p^+}{2}\int\left(\frac{\mu}{6}\right)^2\left(4X^iX^i + \frac{1}{24}\{x^{a_1},\ldots,x^{a_4},X^i\}^2\right) = \frac{p^+}{2}\left(\frac{\mu}{6}\right)^2\int X^i(4+\mathcal{L}_{ab}\mathcal{L}^{ba})X^i .$$

Thus, again the eigenstates will be spherical harmonics on S^5, given explicitly as symmetric traceless polynomials

$$X_l^i = S^i_{a_1\cdots a_l}x^{a_1}\cdots x^{a_l}$$

with corresponding masses

$$M^2 = \left(\frac{\mu}{6}\right)^2 [l(l+4)+4] = \left(\frac{\mu}{6}(l+2)\right)^2 . \tag{B.5}$$

Thus, we get a set of states which are vectors of SO(3) and l-index symmetric traceless tensors of SO(6) with energies $(\mu/6)(l+2)$. These match exactly with the representations **c** in figure 1.

Fermion fluctuations. The quadratic potential for fermions may be determined just as for the case of the supermembrane in [10, section 2]. We start from the superspace M5-brane action in a form valid for coset spaces [19], insert the component field expressions for the superfields (known to all orders for coset spaces), and choose the gauge $\Gamma^+\theta = 0$. In this way, we find a quadratic fermion potential given by

$$V^\psi = -\frac{i}{8}\mu p^+ \Psi^T \gamma^{123} \Psi + \frac{i}{48} \Psi^T \gamma^{ABCD} \{X^A, X^B, X^C, X^D, \Psi\} .$$

Expanding about the spherical fivebrane solution $X^a = r x^a$, $X^i = 0$, we obtain[9]

$$\begin{aligned} V^\psi &= -\frac{i}{8}\mu p^+ \Psi^T \gamma^{123} \Psi + \frac{i\mu}{48} r^4 \Psi^T \gamma^{abcd} \{x^a, x^b, x^c, x^d, \Psi\} \\ &= \frac{\mu}{4} \psi^{\dagger I\alpha} \psi_{I\alpha} - \frac{\mu}{12} \psi^{\dagger I\alpha} g_I^{abJ} \mathcal{L}_{ab} \psi_{J\alpha} . \end{aligned}$$

The normal modes will be eigenstates of the equation

$$g_I^{abJ} \mathcal{L}_{ab} \psi_{J\alpha} = \lambda \psi_{I\alpha}$$

with frequencies given by

$$\omega = \frac{\mu}{4} - \frac{\mu}{12}\lambda . \tag{B.6}$$

Again, these are given in terms of symmetric traceless polynomials in the x^a,

$$\begin{aligned} \psi_I^l &= (\theta_{Ia_1\cdots a_l} + g_I^{ba_1J} \theta_{Jba_2\cdots a_l}) x^{a_1} \cdots x^{a_l} & \lambda &= -l \\ \tilde{\psi}_I^l &= (l\theta_{Ia_1\cdots a_l} + (l+4) g_I^{a_1bJ} \theta_{Jba_2\cdots a_l}) x^{a_1} \cdots x^{a_l} & \lambda &= l+4 , \end{aligned}$$

where θ is totally symmetric and traceless in its SO(6) indices and we have suppressed the SU(2) index.

The two eigenmodes correspond to the two irreducible representations of SO(6) obtained from the tensor product of the symmetric-traceless l-index tensor with a spinor. The modes ψ^l, with energy $\frac{\mu}{6}(l+\frac{3}{2})$ correspond to the irreps with SU(4) Dynkin labels $(1,l,0)$, and these match precisely with the representations **b** in figure 1 (where $n = l+1$). The modes $\tilde{\psi}^l$, with negative frequency $\omega = -\frac{\mu}{6}(l+\frac{1}{2})$ correspond to the irrep with Dynkin label $(0,l-1,1)$. To compare with the matrix model spectrum, we should consider the positive-frequency complex conjugate modes, which have energy $\frac{\mu}{6}(l+\frac{1}{2})$ and lie in the representations with Dynkin label $(1,l-1,0)$. These match exactly with the representations **e** in figure 1.

[9] The conventions used here for fermions are described in [10, appendix A]. In particular, I, J and α, β are SU(4) and SU(2) indices respectively.

Two-form fluctuations. To determine the two-form field fluctuations, we may begin directly with the equation of motion for the two-form field b in a general background, given as [20, equation (9)]. Expanding to quadratic order, we have

$$F_{qrs}g^{sp}\partial_p a = \frac{1}{6\sqrt{-g}} g_{qm}g_{rn}\epsilon^{mnlrsp}\partial_l a F_{rsp}\,,$$

where

$$F_{rsp} = 3\partial_{[r}b_{sp]}\,.$$

Here a is the auxiliary PST scalar and indices $p,q,r,\ldots = 0,\ldots,5$ are covariant worldvolume indices. We may use the gauge symmetries to fix $a = \tau$ and $b_{0\alpha} = 0$ ($\alpha = 1,\ldots,5$). Then the equation of motion becomes

$$\partial_0 b_{\alpha\beta} = \frac{1}{2g^{00}\sqrt{-g}} g_{\alpha\rho}g_{\beta\sigma}\epsilon^{\rho\sigma\mu\nu\lambda}\partial_\mu b_{\nu\lambda}\,.$$

To find the normal modes, we set

$$b_{\alpha\beta} = e^{i\omega t} e^a_\alpha e^b_\beta B_{ab}(x)\,,$$

where $e^a_\alpha \equiv \partial_\alpha x^a$ and B_{ab} may be chosen to satisfy $x^a B_{ab} = 0$. Then the equation of motion gives the eigenvalue equation

$$\frac{\mu}{12}\epsilon^{\mu\nu\rho\sigma\tau} e^a_\mu e^b_\nu e^c_\rho e^d_\sigma \partial_\tau B_{cd} \equiv \frac{\mu}{12}\epsilon^{abcdef}\mathcal{L}_{cd}B_{ef} = i\omega B_{ab}\,.$$

The normal modes may be determined by expanding B_{ab} in terms of traceless symmetric polynomials in x^a and diagonalizing the resulting equation.

We define

$$\begin{aligned}
B^l_{ab} &= (B_{ab;a_1\cdots a_l} - x^a x^c B_{cb;a_1\cdots a_l} + x^b x^c B_{ca;a_1\cdots a_l})x^{a_1}\cdots x^{a_l}\\
\tilde{B}^l_{ab} &= \epsilon_{abcdef}B_{cd;ea_2\cdots a_l}x^{a_2}\cdots x^{a_l}\\
\hat{B}^l_{ab} &= (B_{aa_1;ba_2\cdots a_n} - B_{ba_1;aa_2\cdots a_n} + x^a x^c B_{ba_1;ca_2\cdots a_n} - x^b x^c B_{aa_1;ca_2\cdots a_n})x^{a_1}\cdots x^{a_n}\,,
\end{aligned}$$

where $B_{ab;a_1\cdots a_l}$ is a traceless tensor antisymmetric in a,b and symmetric in $a_1,\ldots,a_l$. Then the eigenmodes are given by

$$\begin{aligned}
B^{l\pm}_{ab} &= \mp\frac{in}{2}B^l_{ab} + (n+2)\tilde{B}^l_{ab} \pm i\hat{B}^l_{ab} & \omega &= \pm\frac{\mu}{6}(l+2)\\
B^{l0}_{ab} &= B^l_{ab} + \hat{B}^l_{ab} & \omega &= 0\,.
\end{aligned}$$

These normal modes correspond to the three irreducible representations in the traceless SO(6) tensor product of the l-index symmetric tensor with the 2- index antisymmetric tensor. The modes $B^{l\pm}$, with energy $\frac{\mu}{6}(l+2)$ lie in the SO(6) representations with Dynkin labels $(2, l-1, 0)$ and $(0, l-1, 2)$. They comprise the positive and negative frequency parts of a set of modes matching exactly with the matrix theory modes **d** in figure 1. The zero-modes correspond to the representation with Dynkin label $(1, l, 1)$, but these are non-physical since they correspond to gauge variations with gauge transformation parameter

$$\Lambda_\alpha = e^a_\alpha B_{ab;a_1\cdots a_n}x^b x^{a_1}\cdots x^{a_n}\,.$$

C. Decoupled 2+1 dimensional field theory from the matrix model

In this appendix, we determine explicitly the form of the commutative 2+1 dimensional field theory arising from the matrix model action expanded about the k-membrane vacuum in the limit where $N \to \infty$ with $(R/\mu)^3/N$ fixed.

We begin by rescaling things so that everything is dimensionless and the quadratic action is independent of the parameters. Then in the limit, we get a continuum theory on a sphere (which we initially take to be a unit sphere) via the following replacements:

$$\begin{aligned}
\mathrm{Tr} &\to \int d\Omega\ \mathrm{tr} \\
M &\to M(\theta,\phi) \\
[J^i, M] &\to L_i M \qquad\qquad L_i \equiv -i\epsilon^{ijk}x^j\partial_k \\
[A,B] &\to \frac{1}{\sqrt{N}}[A,B]\,,
\end{aligned}$$

where the objects on the right hand side are $k \times k$ matrices. Here and below, x^i are the embedding coordinates of the unit sphere, which satisfy $x^2 = 1$. With these replacements, the action becomes

$$S = \int dtd\Omega\ \mathrm{tr}\left(\frac{1}{2}\dot{X}^a\dot{X}^a + \frac{1}{2}\dot{Y}^i\dot{Y}^i + \frac{i}{2}\Psi^\dagger\dot{\Psi} - \frac{1}{2}\frac{1}{36}X^aX^a + \frac{1}{2}\frac{1}{9}(\mathcal{L}_iX^a)^2 - \frac{1}{2}\frac{1}{9}(\mathcal{Y}_i)^2 - \right.$$
$$\left. - \frac{i}{8}\Psi^\dagger\gamma^{123}\Psi\frac{1}{6}\Psi^\dagger\gamma^i\mathcal{L}_i\Psi + \frac{g}{6}\Psi^\dagger\gamma^a[X^a,\Psi] + \frac{1}{36}g^2[X^a,X^b]^2\right).$$

Here, we have defined

$$\begin{aligned}
g &= \left(\frac{R}{\mu}\right)^{3/2}\frac{3}{\sqrt{N}} \\
\mathcal{L}_iA &= L_iA + g[Y^i, A] \\
\mathcal{Y}_i &= Y_i + \frac{i}{2}\epsilon^{ijk}(L_jY_k - L_kY_j + g[Y_j,Y_k])\,.
\end{aligned}$$

The gauge symmetry transforms the fields as

$$\begin{aligned}
\delta Y_i &= L_i\lambda + g[Y^i,\lambda] \\
\delta X_a &= g[X_a,\lambda] \\
\delta\Psi &= g[\Psi,\lambda]
\end{aligned}$$

and $\mathcal{L}$ and $\mathcal{Y}$ are defined to be covariant. The supersymmetry transformation rules are

$$\begin{aligned}
\delta X_a &= -i\epsilon\gamma^a\Psi \\
\delta Y^i &= -i\epsilon\gamma^i\Psi \\
\delta\Psi &= \dot{X}^a\gamma^a\epsilon + \dot{Y}^i\gamma^i\epsilon - \frac{1}{6}X^a\gamma^a\gamma^{123}\epsilon + \frac{i}{6}g[X^a,X^b]\gamma^{ab}\epsilon + \frac{i}{3}\mathcal{L}_iX^a\gamma^{ia}\epsilon + \frac{1}{6}\epsilon^{ijk}\mathcal{Y}^i\gamma^{jk}\epsilon\,.
\end{aligned}$$

The action may then be rewritten by splitting

$$Y^i = x^i \phi + \epsilon^{ijk} x^j A_k \,.$$

It may be checked that

$$A_i = \epsilon^{ijk} Y_j x^k$$

transforms like a conventional gauge field while

$$\phi = x^i Y^i$$

transforms like an adjoint scalar. Note that though A has three components in this notation, these always point tangent to the sphere, so we could rewrite A in terms of a two-component vector field with a worldvolume index. Rescaling the bosonic fields by $\sqrt{3}$, the coupling g by $1/\sqrt{3}$ and the time t by 3 (for convenience), and reintroducing A_0 the resulting action becomes:

$$\begin{aligned} S = \int dt d\Omega \ \mathrm{Tr}\Big(&-\frac{1}{4} F^{\mu\nu} F_{\mu\nu} - \frac{1}{2}(D_\mu X^a)^2 - \frac{1}{2}(D_\mu \phi)^2 + \frac{i}{2}\Psi^\dagger D_0 \Psi - \frac{i}{2}\epsilon^{ijk}\Psi^\dagger \gamma^i x^j D_k \Psi + \\ &+ \frac{g}{2}\Psi^\dagger \gamma^i x^i [\phi, \Psi] + \frac{g}{2}\Psi^\dagger \gamma^a [X^a, \Psi] + \frac{1}{4} g^2 [X_a, X_b]^2 + \frac{1}{2} g^2 [\phi, X^a]^2 - \\ &- \frac{1}{8} X^a X^a - \frac{1}{2}\phi^2 - \frac{3i}{8}\Psi^\dagger \gamma^{123}\Psi + \frac{1}{2}\phi \epsilon^{ijk} x^i F_{jk} \Big) \,. \end{aligned}$$

Here, the first and second lines are just the usual 2+1 dimensional SYM theory. The first line contains the standard kinetic terms for the fields, the second term contains the usual D2-brane interactions, and the final line contains masses for the scalars and fermions and an extra ϕF interaction. The supersymmetry transformation rules may be obtained from the ones above by substituting for Y. Rescaling this action to make the worldvolume a sphere of radius $1/\mu$ gives the desired D2-brane theory (4.2) above.

D. Relationship to the Polchinski-Strassler discussion of $\mathcal{N} = 1^*$ theories

The picture we have presented for the matrix model was inspired by a similar discussion for D3 branes in the context of $\mathcal{N} = 1^*$ theories in [15] . Namely, [15] considered $\mathcal{N} = 4$ Yang-Mills in four dimensions and added a quadratic superpotential that gave a mass to the three chiral multiplets. For weak gauge coupling the vacua are labelled by SU(2) representations. These can be interpreted as D5 branes wrapping an $S^2 \times R^4$. At strong coupling the vacuum with $\phi = 0$ (the trivial SU(2) representation) should be thought of in terms of an expanded spherical NS5-brane. This vacuum is related by S-duality to the vacuum with a single D5 brane.

Here we just point out that this $\mathcal{N} = 1^*$ arises as the DLCQ theory of M-theory on T^3 with a plane wave in the 8 non-compact directions. More explicitly, we can consider the following background in 8 noncompact dimensions

$$\begin{aligned} ds^2 &= -2dx^+ dx^- - |\partial W|^2 (dx^+)^2 + dz_i d\bar{z}_i \\ F_4 &= dx^+ \partial_i \partial_j W \eta^{i\bar{i}} \epsilon_{\bar{i}\bar{l}\bar{m}\bar{k}} d\bar{z}^{\bar{l}} d\bar{z}^{\bar{m}} d\bar{z}^{\bar{k}} dz^j + \text{c.c.} \,, \end{aligned} \tag{D.1}$$

where W is an arbitrary holomorphic function of the three complex coordinates z_i, $i = 1, 2, 3$. This background is related by T and U dualities to the backgrounds considered in [21]. The DLCQ version of this background, with x^- compactified, is expected to be described by $\mathcal{N} = 4$ Yang Mills theory with a superpotential given by W and compactified on T^3.[10] For a quadratic superpotential we obtain the $\mathcal{N} = 1^*$ of [15]. By performing U-dualities on the background described above (D.1) one can obtain backgrounds such that when we put D3 branes on them we get an arbitrary superpotential W on the D3 worldvolume theory. These backgrounds were also studied in [22, 23].

In the context of the DLCQ description of M-theory on T^3, the fact that the transverse M5 is related to an NS5-brane was used in [7] to give some insight on the problem of the transverse M5 branes.

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